DETERMINATION OF THERMAL DIFFUSIVITY FROM EXPERIMENTAL DATA

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An algorithm is discussed for determining the temperature dependence of the thermal diffusivity from thermocouple measurements at one or more points within a body.

Various stationary and nonstationary methods have been developed [1] for determining the temperature dependence of the thermophysical characteristics of materials. Most of the existing methods are based on an analytic solution of very simple heat-conduction problems, which imposes certain restrictions on their use. Stationary methods are very time consuming, since a separate experiment is generally required to obtain a single point on the curve for a thermophysical coefficient as a function of temperature. One nonstationary experiment can yield values of thermophysical properties over a wide range of temperatures.

The most promising methods for determining thermophysical properties are methods based on solutions of inverse nonstationary heat-conduction problems involving the thermophysical coefficients. In this case the temperature dependence of the thermophysical characteristics of a material is determined by using known boundary conditions and measured values of the temperature inside a body.

We consider the numerical determination of the polynomial dependence of the thermal diffusivity $a(T) = \sum_{k=0}^{N} a_k T^k$ from the conditions

$$\frac{\partial T}{\partial \tau} = a(T)\frac{\partial^2 T}{\partial x^2}, \quad 0<x<b, \quad 0<\tau\leqslant\tau_m, \tag{1}$$

$$T(0, \tau) = q_1(\tau) \quad \text{or} \quad -\frac{\partial T(0, x)}{\partial x} = q_1(\tau), \tag{2}$$

$$T(b, \tau) = q_2(\tau) \quad \text{or} \quad -\frac{\partial T(b, \tau)}{\partial x} = q_2(\tau), \tag{3}$$

$$T(x, 0) = \varphi(x), \quad 0\leqslant x\leqslant b, \tag{4}$$

$$T(x_p, \tau) = f_p(\tau), \quad p = 1, 2, \ldots, M, \quad 0\leqslant x_p\leqslant b, \tag{5}$$

where $q_1(\tau)$, $q_2(\tau)$, $\varphi(x)$, and $f_p(\tau)$ are known functions.

The criterion for choosing the unknown parameters a_k, k = 0, 1, ..., N is written in the form

$$I(a_0, a_1, \ldots, a_N) = \sum_{p=1}^{M}\int_0^{\tau_m}[T_p(\tau) - f_p(\tau)]^2 d\tau \to \min, \tag{6}$$

where the $T_p(\tau)$ are the calculated values of the temperature at the points x_p, p = 1, 2, ..., M where the thermocouples are located, and the $f_p(\tau)$ are the measured temperatures at these same points.

The problem formulated is the simplest inverse problem involving the thermophysical coefficients

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and is of interest largely as an illustration of the method. In practice this formulation can be used to determine the volumetric heat capacity of a material $C(T)$ for a known thermal conductivity $\lambda(T)$. In this case the heating problem is reduced to the form (1)-(4) by the Kirchhoff substitution

$$y = \frac{1}{\lambda_0} \int_0^T \lambda(\xi)\, d\xi$$

The heating of a sample in the extremal problem (1)-(6) can be realized for boundary conditions which vary arbitrarily with time.

We use the gradient method to minimize the functional (6). We calculate the derivatives of the functional with respect to the parameters sought for,

$$\frac{\partial I}{\partial a_k} = 2 \sum_{p=1}^{M} \int_0^{\tau_m} [T_p(\tau) - f_p(\tau)] \frac{\partial T_p(\tau)}{\partial a_k} d\tau, \quad k = 0, 1, \ldots, N. \tag{7}$$

Equation (7) involves $N + 1$ unknown functions $\Theta_k = [\partial T_p(\tau)/\partial a_k]$, $k = 0, 1, \ldots, N$. To determine them we differentiate Eqs. (1)-(4) with respect to a_k. We obtain $N + 1$ boundary-value problems

$$\frac{\partial \Theta_k}{\partial \tau} = \left(\sum_{l=0}^{N} a_l T^l\right) \frac{\partial^2 \Theta}{\partial x^2} + \left(\sum_{l=1}^{N} l a_l T^{l-1}\right) \frac{\partial^2 T}{\partial x^2} \Theta_k + T^k \frac{\partial^2 T}{\partial x^2}, \tag{8}$$

$$\begin{gathered} 0 < x < b, \quad 0 < \tau \leqslant \tau_m, \\ \Theta_k(x, 0) = 0, \quad 0 \leqslant x \leqslant b, \\ \Theta_k(0, \tau) = \Theta_k(b, \tau) = 0 \end{gathered} \tag{9}$$

or

$$\begin{gathered} \frac{\partial \Theta_k(0, \tau)}{\partial x} = \frac{\partial \Theta_k(b, \tau)}{\partial x} = 0, \\ k = 0, 1, \ldots, N. \end{gathered} \tag{10}$$

We use the method of steepest descent, constructing the approximations by using the relation

$$a_k^{r+1} = a_k^r - \beta_r \frac{\partial I}{\partial a_k}, \quad k = 0, 1, \ldots, N, \tag{11}$$

where r is the number of the iteration.

The descent parameter β_r is determined from the condition that the functional (6) be minimum at the $(r + 1)$-th iteration,

$$\min_{\beta_r} I\left(a_k^r - \beta_r \frac{\partial I}{\partial a_k}, \; k = 0, 1, \ldots, N\right).$$

Parameter β_r can be chosen explicitly. Suppose a_k^r ($k = 0, 1, \ldots, N$) is changed by $-\beta_r(\partial I/\partial a_k)$. Then the function $T(x, \tau)$ changes by $\vartheta_r(x, \tau)$. Neglecting second-order quantities we obtain from Eqs. (1)-(4)

$$\frac{\partial \vartheta_r}{\partial \tau} = a(T) \frac{\partial^2 \vartheta_r}{\partial x^2} - \beta_r \left(\sum_{k=0}^{N} \frac{\partial I}{\partial a_k} T^k\right) \frac{\partial^2 T}{\partial x^2}, \tag{12}$$

$$\begin{gathered} 0 < x < b, \quad 0 < \tau \leqslant \tau_m, \\ \vartheta_r(x, 0) = 0, \quad 0 \leqslant x \leqslant b, \end{gathered} \tag{13}$$

$$\vartheta_r(0, \tau) = \vartheta_r(b, \tau) = 0 \quad \text{or} \quad \frac{\partial \vartheta_r(0, \tau)}{\partial x} = \frac{\partial \vartheta_r(b, \tau)}{\partial x} = 0. \tag{14}$$

Problem (12)-(14) is linear in β_r. Therefore we can write

$$I^{r+1} = \sum_{p=1}^{M} \int_0^{\tau_m} [T_p(\tau) - f_p(\tau) - \beta_r \vartheta_r(x_p, \tau)]^2 d\tau.$$

TABLE 1. Values of the Functional I and the Coefficients a_0 and a_1 Determining the Polynomial

	0	1	...	30	...	55
I	$0{,}469\cdot10^{-1}$	$0{,}1826\cdot10^{-3}$	...	$0{,}4616\cdot10^{-7}$	...	$0{,}7628\cdot10^{-10}$
a_0	0,5	0,8567	...	0,8011	...	0,79999
a_1	0,2	0,2726	...	0,3960	...	0,39999

Hence, since it is necessary that $\partial I^{r+1}/\partial \beta_r = 0$, we obtain

$$\frac{\sum_{p=1}^{M}\int_0^{\tau_m}[T_p(\tau)-f_p(\tau)]\,\vartheta_r(x_p,\tau)\,d\tau}{\sum_{p=1}^{M}\int_0^{\tau_m}[\vartheta_r(x_p,\tau)]^2\,d\tau}. \tag{15}$$

The iterative process is constructed in the following way: The zero approximation of the required parameters is specified and problem (1)-(4) is solved. Using the calculated temperature distribution problem (8)-(10) is solved N + 1 times and the gradient of the functional is calculated from Eq. (7). The function $\vartheta_r(x,\tau)$ is determined from (12)-(14) and the depth of descent β_r is calculated. After this the new approximation is found from Eq. (11) and used for the next iteration.

The algorithm described above was programmed in ALGOL for a BÉSM-6 computer. An implicit boundary-value problem approximation scheme was used with the net

$$\omega = \{x_i = hi,\ i = 0,\ 1,\ \ldots,\ n;\ \tau_j = \Delta\tau j,\ j = 0,\ 1,\ \ldots,\ m\}$$

(cf. [2]).

The results of a calculation to illustrate the method are shown in Table 1. For an a priori known function $a(T) = 0.8 + 0.4\,T$ the temperature distribution was obtained in an infinite plate of thickness b = 1 for the following boundary conditions:

$$\varphi(x) = 0 = \text{const},\ \frac{\partial T(0,\tau)}{\partial x} = 1,\ \frac{\partial T(b,\tau)}{\partial x} = 0.$$

The temperature at x = b was used as input data for the regeneration of $a(T)$. The calculation was performed for an $n \times m = 50 \times 50$ net and required about 15 min.

No particular difficulty is involved in extending the present method to a region with moving boundaries. Similarly an algorithm can be constructed to determine another coefficient in the heat-conduction equation.

It should be noted that the a priori specification of the degree of the regenerating polynomial is not a particularly stringent restriction.

The results of processing the experimental data show that the temperature dependence of the thermophysical characteristics of various materials is adequately approximated by polynomials of no higher degree than the third.

Criterion (6) can be used when the input temperatures are known exactly. If the input data are in error it is necessary to use the principle of discrepancy

$$I = \sum_{p=1}^{M}\int_0^{\tau_m}[T_p(\tau)-f_p(\tau)]^2 d\tau \approx \delta, \tag{16}$$

where

$$\delta = \sum_{p=1}^{M}\int_0^{\tau_m}\sigma_p(\tau),\ d\tau;$$

$\sigma_p(\tau)$ is the mean square deviation of the input temperatures at the points $x = x_p$. We note that in a real experiment the degree of the polynomial sought for can also be chosen from condition (16).

NOTATION

$a(T)$, thermal diffusivity; T, temperature; x, coordinate; τ, time; $\varphi(x)$, initial temperature distribution; a_k, polynomial coefficient; $f_p(\tau)$, input temperatures; y, model temperature; $\lambda(T)$, thermal conductivity; I, functional; δ, error of input data; q, heat flux or temperature on the boundary of the region.

LITERATURE CITED

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